Geometric equivalence among smooth map-germs associated with representations of Lie groups (Singularity theory of differential maps and its applications)

Izumiya, Shyuichi

数理解析研究所講究録
第2049巻 2017年 83-93

Geometric equivalence among smooth map-germs associated with representations of Lie groups

北海道大学・大学院理学研究院・数学部門　泉屋　周一
Shyuichi Izumiya
Department of Mathematics, Hokkaido University

1 Introduction

Let G be a real Lie group and $\rho : G \longrightarrow GL(p,\mathbb{R})$ a smooth representation (i.e a homomorphism and a smooth mapping). For map-germs $f,g : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$, we say that f and g are $\mathcal{K}[\rho(G)]$-*equivalent* if there exist a diffeomorphism germ $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ and a smooth map germ $M : (\mathbb{R}^n,0) \longrightarrow (G,M(0))$ such that $f\circ\phi(x) = \rho(M(x))g(x)$, for any $x \in (\mathbb{R}^n,0)$. We also say that f,g are $\mathcal{C}[\rho(G)]$-*equivalent* if $\phi = 1_{\mathbb{R}^n}$. In the case when G is a Lie subgroup of $GL(p,\mathbb{R})$, the above equivalence is exactly the same as G-equivalence introduced by Tougeron [17]. In [6, 7, 8], Gervais investigated the basic properties of G-equivalence. Tougeron and Gervais mentioned that there might be several examples of G-equivalence depending on G. However, they only gave the above two cases as examples in their contexts. After the papers of Gervais appeared, Damon has published paper which give a quite general framework for the theory of singulairties of smooth map germs [1]. Since G-equivalence is included in the framework of Damon, nobody has investigated it until now. One of the reasons why the notion of G-equivalence has not been paid attention is that there have been no interesting examples. If G is connected , then $\rho(G)$ is a Lie subgroup of $GL(p,\mathbb{R})$ by the theorem of Yamabe, so that the basic frameworks for the above equivalence follows from those in [6, 7, 8]. If we adopt $G = GL(p,\mathbb{R})$ and $\rho = 1_{GL(p,\mathbb{R})}$, then $\mathcal{K}[G]$-equivalence is $\mathcal{K}$-equivalence and $\mathcal{C}[G]$-equivalence is $\mathcal{C}$-equivalence in [15] respectively. If $G = \{I\} \subset GL(p,\mathbb{R})$ and $\rho = \iota$ is the inclusion, then $\mathcal{K}[G]$-equivalence is $\mathcal{R}$-equivalence. In [6, 7, 8] Gervais only mentioned these cases as examples of G-equivalence. In this paper we give several interesting examples of $\mathcal{K}[\rho(G)]$-equivalence. In particular, there exist applications to quantum chemistry and spintronics in [16]. The representations of Lie groups are essentially needed for those examples.

On the other hand, we consider other equivalence relations among map-germs. For map-germs $f,g : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$, we say that f and g are $\mathcal{A}[\rho(G)]$-*equivalent* if there exist diffeomorphism germs $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ and $\psi : (\mathbb{R}^p,0) \longrightarrow (\mathbb{R}^p,0)$ such that $J_\psi(y) \in \rho(G)$ for any $y \in (\mathbb{R}^p,0)$ and $f\circ\phi = \psi\circ g$. Here $J_\psi(y)$ is the Jacobian matrix of ψ at $y \in (\mathbb{R}^p,0)$. We also say that f,g are $\mathcal{L}_J[\rho(G)]$-*equivalent* if $\phi = 1_{\mathbb{R}^n}$. We remark that if $G = \{I\}$, then $\mathcal{A}[I]$-equivalence is $\mathcal{R}$-equivalence. Moreover, $\mathcal{A}[GL(p,\mathbb{R})]$-equivalence is $\mathcal{A}$-equivalence. However, $\mathcal{A}[\rho(G)]$-equivalence does not imply $\mathcal{K}[\rho(G)]$-equivalence, generally.

For a diffeomorphism germ $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$, we can show that there exist function-germs $\psi_{ij} : (\mathbb{R}^p, 0) \longrightarrow \mathbb{R}$, $(i, j = 1, \dots, p)$ such that $(\psi_{ij}(y)) \in GL(p, \mathbb{R})$ and

$$\psi(y) = (\sum_{j=1}^{p} \psi_{1j}(y)y_j, \dots, \sum_{j=1}^{p} \psi_{pj}(y)y_j)$$

for $y = (y_1, \dots, y_p) \in (\mathbb{R}^p, 0)$. We say that ψ is a *$\rho(G)$-diffeomorphism* if $(\psi_{ij}(y)) \in \rho(G)$. For map-germs $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, we say that f and g are *$\mathcal{A}^*[\rho(G)]$-equivalent* if there exist a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and a $\rho(G)$-diffeomorphism germ $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$ such that $f \circ \phi = \psi \circ g$. We also say that f, g are *$\mathcal{L}^*[\rho(G)]$-equivalent* if $\phi = 1_{\mathbb{R}^n}$. We remark that if $G = \{I\}$ and $\rho = \iota$ is the inclusion, then $\mathcal{A}^*[I]$-equivalence is $\mathcal{R}$-equivalence. Moreover, $\mathcal{A}^*[GL(p, \mathbb{R})]$-equivalence is $\mathcal{A}$-equivalence in [15]. By definition, if f, g are $\mathcal{A}^*[\rho(G)]$-equivalent, then these are $\mathcal{K}[\rho(G)]$-equivalent. We remark that $\mathcal{A}^*[GL(p, \mathbb{R})]$-equivalence always induces a geometric subgroup of $\mathcal{A}$ and $\mathcal{K}$ in the sense of Damon [1] (cf. §4). However, there are no good examples of such equivalence except trivial cases (i.e. $\mathcal{R}$ and $\mathcal{A}$) so far.

We can also consider the following mixed equivalence of the above equivalence relations. We consider pairs of map-germs $(f_1, f_2) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p \times \mathbb{R}^q, (0, 0))$ and two representations of Lie groups $\rho_1 : G_1 \longrightarrow GL(p, \mathbb{R}), \rho_2 : G_2 \longrightarrow GL(q, \mathbb{R})$. We say that (f_1, f_2) and (g_1, g_2) are *$(\mathcal{K}[\rho_1(G_1)], \mathcal{A}[\rho_2(G_2)])$-equivalent* if there exist a diffeomorphism-germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$, a map-germ $M : (\mathbb{R}^n, 0) \longrightarrow (\rho_1(G), M(0))$ and a diffeomorphism germ $\psi : (\mathbb{R}^q, 0) \longrightarrow (\mathbb{R}^q, 0)$ such that $J_\psi(y_2) \in \rho_2(G_2)$, $f_1 \circ \phi(x) = M(x)g_1(x)$ and $f_2 \circ \phi(x) = \psi \circ g_2(x)$ for any $(x, y_2) \in (\mathbb{R}^n \times \mathbb{R}^q, (0, 0))$. However, we do not give detailed descriptions here.

In [11] the basic framework for the study of $\mathcal{K}[\rho(G)]$-equivalence and $\mathcal{A}[\rho(G)]$-equivalence. In this paper we only give several examples of these equivalece. This paper depends on the joint project concerning on applications of singularity theory to quantum physics and chemistry with Masatomo Takahashi and Hiroshi Teramoto.

We assume that all map-germs and manifolds are class C^∞ unless stated otherwise.

2 Examples of $\mathcal{K}[\rho(G)]$-equivalence I: $\iota : G \hookrightarrow GL(p, \mathbb{R})$

In this section we consider examples of $\mathcal{K}[\rho(G)]$-equivalence for $G \subset GL(p, \mathbb{R})$ and $\rho = \iota : G \hookrightarrow GL(p, \mathbb{R})$ is the inclusion map. Actually $\mathcal{K}[\rho(G)]$-equivalence is G-equivalence in the sense of Tougeron [17]. Even in this case, there are several nontrivial important examples. The detailed investigations for these examples will be appeared in elsewhere.

Example 2.1 (Flags of varieties) We consider that

$$G = T^*(p) = \left\{ A = \begin{pmatrix} \lambda_{11} & 0 & \dots & 0 \\ \lambda_{21} & \lambda_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{p1} & \lambda_{p2} & \dots & \lambda_{pp} \end{pmatrix} \;\middle|\; \lambda_i \in \mathbb{R}, i = 1, 2, \lambda_{11}\lambda_{22} \cdots \lambda_{pp} \neq 0 \right\}$$

and $\iota : T^*(p) \hookrightarrow GL(p, \mathbb{R})$ is the inclusion map. For $f = (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, define $V_j(f) = \cap_{i=1}^{i=j} f_i^{-1}(0)$, $j = 1, \dots p$, so that we have $V_1(f) \supset V_2(f) \supset \dots \supset V_p(f)$. Then a *flag variety* with respect to f is defined to be

$$FV(f) = (V_1(f), V_2(f), \dots, V_p(f)).$$

For $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, we say that $FV(f), FV(g)$ are *flag $\mathcal{K}$-equivalent* if f, g are $\mathcal{K}[T^*(p)]$-equivalent. By definition, if $FV(f), FV(g)$ are flag $\mathcal{K}$-equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $\phi(V_j(f)) = (V_j(g))$ as set germs for $j = 1, \dots, p$.

Example 2.2 (Hypersurface arrangements)

$$G = D^*(p) = \left\{ A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_p \end{pmatrix} \;\middle|\; \lambda_i \in \mathbb{R}, i = 1, 2, \lambda_1\lambda_2 \cdots \lambda_p \neq 0 \right\}$$

and $\rho : D^*(p) \subset GL(p, \mathbb{R})$ is the inclusion map. For $f = (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, define $V(f_i) = f_i^{-1}(0)$, $i = 1, \dots p$. We call $AV(f) = (V(f_1), \dots V(f_p)))$ a hypersurface arrangememt. We say that $AV(f), AV(g)$ are *arrangement $\mathcal{K}$-equivalent* if f, g are $\mathcal{K}[D^*(p)]$-equivalent. By definition, if $AV(f), AV(g)$ are arrangement $\mathcal{K}$-equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $\phi(V(f_i)) = (V(g_i))$ as set germs for $i = 1, \dots, p$.

For $f_1(x_1, x_2) = x_1, f_2(x_1, x_2) = x_1^2 - x_2^3$ and $g_1(x_1, x_2) = x_2, g_2(x_1, x_2) = x_1^3 - x_2^4$, $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are not $\mathcal{K}[D^*(2)]$-equivalent (i.e. $AV(f)$ and $AV(g)$ are not arrangement $\mathcal{K}$-equivalent) but $\mathcal{K}$-equivalent.

Example 2.3 (Functions on varieties)

$$(1) \qquad G = \{1\} \oplus GL(p-1, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in GL(p-1, \mathbb{R}) \right\},$$

$$(2) \qquad G = (1^+, GL(p-1, \mathbb{R})) = \left\{ \begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix} \;\middle|\; A \in GL(p-1, \mathbb{R}) \right\}.$$

In the case of (1), for $f = (f_1, \dots, f_p), g = (g_1, \dots, g_p) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, f, g are $\mathcal{K}[\{1\} \oplus GL(p-1, \mathbb{R})]$-equivalent if and only if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $f_1 \circ \phi = g_1$ and

$$\langle f_2 \circ \phi, \dots, f_p \circ \phi \rangle_{\mathcal{E}_{p-1}} = \langle g_2, \dots, g_p \rangle_{\mathcal{E}_{p-1}}.$$

We define a variety $V(f_2, \dots, f_p) = \cap_{i=2}^p f_i^{-1}(0)$. If f, g are $\mathcal{K}[\{1\} \oplus GL(p-1, \mathbb{R})]$-equivalent, then there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that $f_1 \circ \phi = g_1$ and $\phi(V(g_2, \dots, g_p)) = V(f_2, \dots, f_p)$ as set germs. Moreover, we consider the case (2). Then f, g are $\mathcal{K}[(1^+, GL(p-1, \mathbb{R}))]$-equivalent if and only if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ such that

$$f_1 \circ \phi - g_1 \in \langle f_2 \circ \phi, \dots, f_p \circ \phi \rangle_{\mathcal{E}_{p-1}} \text{ and } \langle f_2 \circ \phi, \dots, f_p \circ \phi \rangle_{\mathcal{E}_{p-1}} = \langle g_2, \dots, g_p \rangle_{\mathcal{E}_{p-1}}.$$

It follows that $\phi(V(g_2,\dots,g_p)) = V(f_2,\dots,f_p)$ and $f_1\circ\phi|_{V(g_2,\dots,g_p)} = g_1|_{V(g_2,\dots,g_p)}$.

We consider $(f,h),(g,h) : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}\times\mathbb{R}^{p-1},(0,0))$. We say that $(f,h),(g,h)$ are $\mathcal{R}_{I(h)}$-*equivalent* if there exists a diffeomorphism-germ $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $f = g\circ\phi$ and $\langle h_1\circ\phi,\dots,h_{p-1}\circ\phi\rangle_{\mathcal{E}_n} = \langle h_1,\dots,h_{p-1}\rangle_{\mathcal{E}_n}$. In [10, Lemma 1.3], it has been shown that $(f,h),(g,h)$ are $\mathcal{R}_{I(h)}$-equivalent if and only if $(f,h),(g,h)$ are $\mathcal{K}[\{1\}\oplus GL(p-1,\mathbb{R})]$-equivalent. Moreover, $\mathcal{K}[\{1\}\oplus GL(q,\mathbb{R})]$-equivalence was also investigated in [2].

We also consider

$$G = (1^+, GL(q,\mathbb{R})) = \left\{ \begin{pmatrix} 1 & {}^t\boldsymbol{b} \\ 0 & A \end{pmatrix} \;\middle|\; \boldsymbol{b}\in\mathbb{R}^q,\ A\in GL(q,\mathbb{R}) \right\} \subset GL(1+q,\mathbb{R}).$$

Then f,g are $\mathcal{K}[(1^+,GL(q,\mathbb{R}))]$-equivalent if and only if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that

$$f_1\circ\phi - g_1 \in I(f_2\circ\phi) \text{ and } I(f_2\circ\phi) = I(g_2),$$

where $I(f_2) = f_2^*(\mathfrak{M}_q)\mathcal{E}_n$. It follows that $\phi(V(g_2)) = V(f_2)$ and $f_1\circ\phi|_{V(g_2)} = g_1|_{V(g_2)}$. For $(f,h),(g,h) : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}\times\mathbb{R}^{p-1},(0,0))$, $(f,h),(g,h)$ are $\mathcal{K}[(1^+,GL(q,\mathbb{R}))]$-equivalent if and only if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $f\circ\phi - g \in I(h\circ\phi)$ and $I(h\circ\phi) = I(h)$. In this case we have $\phi(V(h)) = V(h)$ and $f\circ\phi|_{V(h)} = g|_{V(h)}$. Therefore, $\mathcal{R}_{I(h)}$-equivalence can be regarded as $\mathcal{R}$-equivalence among function germs around a fixed variety $V(h)$ which is described by $\mathcal{K}[\{1\}\oplus GL(q,\mathbb{R})]$-equivalence. On the other hand, $\mathcal{K}[(1^+,GL(q,\mathbb{R}))]$-equivalence induces $\mathcal{R}$-equivalence among function germs on a fixed variety $V(h)$.

3 Examples of $\mathcal{K}[\rho(G)]$-equivalence II: $\rho : G \longrightarrow GL(p,\mathbb{R})$

In this section we give two important applications when the representation $\rho : G \longrightarrow GL(p,\mathbb{R})$ is definitely needed.

Example 3.1 (Traceless Hermitian matrices) We consider the special unitary group $G = SU(m)$ and the set of traceless Hermitian matrices:

$$\mathrm{Herm}_0(m) = \{ X\in M_m(\mathbb{C}) \mid X^* = X, \mathrm{Trace}\, X = 0 \},$$

where X^* is the adjoint matrix of X. The Lie algebra of $SU(m)$ is the set of traceless Hermitian anti-symmetric matrices:

$$\mathfrak{su}(m) = \{ X\in M_m(\mathbb{C}) \mid X^* = -X, \mathrm{Trace}\, X = 0 \}.$$

Both of $\mathfrak{su}(m)$ and $\mathrm{Herm}_0(m)$ are $\mathbb{R}$-vector spaces with m^2-1-dimesion. It is easy to show that $X = A+iB \in \mathrm{Herm}_0(m)$ if and only if $-iX = B-iA \in \mathfrak{su}(m)$ for $A,B\in M_n(m,\mathbb{R})$. We define a mapping $\iota : \mathfrak{su}(m) \longrightarrow \mathrm{Herm}_0(m)$ by $\iota(X) = -iX$. Then we have $\iota^{-1}(Y) = iY$. Moreover, it is known that there is a positive definite scaler product on $\mathfrak{su}(m)$ defined by $\langle X,Y\rangle = \mathrm{Trace}\, XY^*$. We can also define a a positive definite scaler product on $\mathrm{Herm}_0(m)$ by $(X,Y) = \mathrm{Trace} XY^*$. Then we have $(\iota X,\iota Y) = \mathrm{Trace}\,(-iX)(-iY)^* = \mathrm{Trace}\,(-iX(iY^*)) =$

Trace $XY^* = \langle X, Y\rangle$, so that ι is an isometry. Since Trace $AXA^* =$ Trace X for $A \in SU(m)$ and $X \in M_m(\mathbb{C})$, we have $AXA^* \in \mathrm{Herm}_0(m)$ for $A \in SU(m)$ and $X \in \mathrm{Herm}_0(m)$. We now define the adjoint representation $Ad : SU(m) \longrightarrow \mathrm{Iso}(\mathrm{Herm}_0(m))$ by $Ad(A)(X) = AXA^*$ for $X \in \mathrm{Herm}_0(m)$, where Iso $(\mathrm{Herm}_0(m))$ is the group of isometry over $\mathrm{Herm}_0(m)$. For $p = m^2-1$, let $\Sigma = \{\sigma_1, \dots \sigma_p\}$ be an orthonormal basis of $\mathrm{Herm}_0(m)$ with respect to $(\ , \ )$. We now fix the orthonormal basis $\Sigma = \{\sigma_1, \dots \sigma_p\}$ of $\mathrm{Herm}_0(m)$, then we have an isometry $\phi_\Sigma : \mathrm{Herm}_0(m) \longrightarrow \mathbb{R}^p$ defined by $\phi_\Sigma(\sum_{i=1}^p x_i\sigma_i) = {}^t(x_1, \dots, x_p) = x$, where $\mathbb{R}^p$ is the Euclidean space with the canonical scaler product. Therefore we have the canonical identification of Iso $(\mathrm{Herm}_0(m))$ with $SO(p)$ depending on the orthonormal basis $\Sigma = \{\sigma_1, \dots \sigma_p\}$. By using the isometry ϕ_Σ, we have a Lie group homomorphism $\rho_\Sigma : SU(m) \longrightarrow SO(p)$ defined by $\rho_\Sigma(A)(y) = \phi_\Sigma \circ Ad(A) \circ \phi_\Sigma^{-1}(y)$ for any $A \in SU(m)$ and $y \in \mathbb{R}^p$.

On the other hand, let $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathrm{Herm}_0(m), O)$ be C^∞-map germs. We say that f, g are *$\mathcal{SU}(m)$-equivalent* if there exists a map germ $A : (\mathbb{R}^n, 0) \longrightarrow SU(m)$ and a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n)$ such that $f \circ \phi(x) = A(x)g(x)A^*(x)$ for $x \in (\mathbb{R}^n, 0)$. Then we have the following proposition.

Proposition 3.2 *Let $\Sigma = \{\sigma_1, \dots \sigma_p\}$ be an orthonormal basis of* $\mathrm{Herm}_0(m)$ *with respect to $(\ , \ )$, where $p = m^2 - 1$. Then $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathrm{Herm}_0(m), O)$ are $\mathcal{SU}(m)$-equivalent if and only if $\phi_\Sigma \circ f, \phi_\Sigma \circ g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ are $\mathcal{K}[\rho_\Sigma(SU(m))]$-equivalent.*

In [16], the case $m = 2$ (i.e. $p = 3$) has been considered. For any $H \in \mathrm{Herm}_0(2)$, there exist $h_i \in \mathbb{R}$, $(i = 1, 2, 3)$, such that

$$H = \sqrt{2}\begin{pmatrix} h_3 & h_1 - ih_2 \\ h_1 + ih_2 & -h_3 \end{pmatrix}.$$

In this case we have Pauli matrices defined by

$$\overline{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \overline{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \overline{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are 2×2-traceless Hermitian matrices. We can show that $(\overline{\sigma}_i, \overline{\sigma}_j) = 2\delta_{ij}$, so that $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ is an orthonormal basis of $\mathrm{Herm}_0(2)$, where $\sigma_i = \overline{\sigma}_i/\sqrt{2}$. Then we have $H = h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$. It follows that we have $\rho_\Sigma : SU(2) \longrightarrow SO(3)$. On there other hand, $-i\Sigma = \{\delta_1 = -i\sigma_1, \delta_2 = -i\sigma_2, \delta_3 = -i\sigma\}$ is an orthonormal basis of $\mathfrak{su}(2)$. Then we also have the adjoint map $\overline{Ad} : SU(2) \longrightarrow SO(3)$ defined by $\overline{Ad}(A)(Y) = AYA^*$ for $Y \in \mathfrak{su}(2)$. It is classically known (cf.[19]) that $\rho_{-i\Sigma}(SU(2)) = SO(3)$. This fact is known that $Spin(3) \cong SU(2)$ (cf. [12]). For any $A \in SU(2)$, we have $\iota \circ \overline{Ad}(A) = Ad(A) \circ \iota$, so that $\rho_\Sigma(SU(2)) = SO(3)$. Therefore, for any $f : (\mathbb{R}^n, 0) \longrightarrow (\mathrm{Herm}_0(2), O)$, there exist $f_i : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$, $(i = 1, 2, 3)$, such that $f(x) = f_1(x)\sigma_1 + f_2(x)\sigma_2 + f_2(x)\sigma_3$. Then $\phi_\Sigma \circ f(x) = (f_1(x), f_2(x), f_3(x))$. By Proposition 2.1, $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathrm{Herm}_0(2), O)$ are $\mathcal{SU}(2)$-equivalent if and only $\phi_\Sigma \circ f, \phi_\Sigma \circ g$ are $\mathcal{K}[SO(3)]$-equivalent. We emphasize that f can be considered as a quantum mechanical Hamiltonian for energy levels crossing problems appearing spintronics and quantum chemistry, etc [9]. In [16], a classification of $f : (\mathbb{R}^3, 0) \longrightarrow (\mathrm{Herm}_0(2), O)$ by $\mathcal{SU}(2)$-equivalence is given. The detailed arguments will be appeared in elsewhere.

On the other hand, there is another important quantum mechanical Hamiltonian in [9] as follows:

$$H(x)=\begin{pmatrix} f_1(x) & 0 & f_2(x)+if_3(x) & f_4(x)+if_5(x)\\ 0 & f_1(x) & -f_4(x)+if_5(x) & f_2(x)-if_3(x)\\ f_2(x)-if_3(x) & -f_4(x)-if_5(x) & -f_1(x) & 0\\ f_4(x)-if_5(x) & f_2(x)+if_3(x) & 0 & -f_1(x)\end{pmatrix},$$

where $f_i : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$, $i = 1,2,3,4,5$, are function-germs. We remark that this is a 4×4-traceless Hermitian matrix. This Hammiltonian matrix is called *Type J* in [9]. We have the canonical embedding $\iota : SU(2)\times SU(2) \longrightarrow SU(4)$ defined by $\iota(A_1,A_2) = A_1 \oplus A_2$, so that $\mathcal{SU}(4)$-equivalence among $f : (\mathbb{R}^n,0) \longrightarrow \mathrm{Herm}_0(4)$ induces $\mathcal{SU}(2)\times\mathcal{SU}(2)$-equivalence among $f : (\mathbb{R}^n,0) \longrightarrow \mathrm{Herm}_0(4)$. The set of the matrices of the above form is a $\mathbb{R}$-linear subspace of $\mathrm{Herm}_0(4)$, so that we now determine this space. Our matrix has the form

$$H=\begin{pmatrix} x_1 & 0 & x_2+ix_3 & x_4+ix_5\\ 0 & x_1 & -x_4+ix_5 & x_2-ix_3\\ x_2-ix_3 & -x_4-ix_5 & -x_1 & 0\\ x_4-ix_5 & x_2+ix_3 & 0 & -x_1\end{pmatrix}, \qquad (*)$$

for $x_i \in \mathbb{R}$, $i = 1,2,3,4,5$. Let $H_J(4)$ be the set of Hermitaian 4×4-matrices of the above form. Then we can show that $H_J(4)$ is an $\mathbb{R}$-linear subspace of $\mathrm{Herm}_0(4)$ such that $\dim_{\mathbb{R}} H_J(4) = 5$. Moreover, for a Hamiltonian $H(x)$ of Type J, we have a map-germ $f : (\mathbb{R}^n,0) \longrightarrow H_J(4)$. Then we can show that $\mathcal{SU}(2)\times\mathcal{SU}(2)$-equivalence among map-germs $f : (\mathbb{R}^n,0) \longrightarrow H_J(4)$ is well-defined. It is also known that $SU(2)\times SU(2) \cong Spin(4)$ (cf. [12]). Let $\mathbb{H}$ be the skew field of quaternions. Then we have a representation of $SU(2)\times SU(2)$ defined by the linear isomorphism $X \mapsto A_1XA_2^*$ for $(A_1,A_2) \in SU(2)\times SU(2)$ and $X \in \mathbb{H}$. Since there is an isometry $\mathbb{H} \cong \mathbb{R}^4$, we have a representation $\rho : SU(2)\times SU(2) \longrightarrow GL(4,\mathbb{R})$ such that $\rho(SU(2)\times SU(2)) = SO(4)$. Moreover, this representation is a double covering over $SO(4)$. We can show that there exists an isometry $\psi : H_J(4) \longrightarrow \mathbb{H}$ and show the following proposition:

Proposition 3.3 *For $f,g : (\mathbb{R}^n,0) \longrightarrow H_J(4)$, f,g are $\mathcal{SU}(2)\times\mathcal{SU}(2)$-equivalent if and only if $\psi\circ f, \psi\circ g$ are $\mathcal{K}[\{1\}\oplus SO(4)]$-equivalent.*

We will give a classification of map germs $f : (\mathbb{R}^n,0) \longrightarrow H_J(4)$ by $\mathcal{SU}(2)\times\mathcal{SU}(2)$-equivalence in the forthcoming paper.

Example 3.4 (Traceless real symmetric matrices) We consider the special orthogonal group $G = SO(m)$ and the set of traceless real symmetric matrices:

$$\mathrm{Sym}_0(m) = \{X \in M_m(\mathbb{R}) \mid X = {}^tX, \mathrm{Trace}\, X = 0\}.$$

In this case $\dim \mathrm{Sym}_0(m) = \frac{m(m+1)}{2} - 1$. We also have the positive definite scaler product $(X,Y) = \mathrm{Trace}\, XY$ for $X,Y \in \mathrm{Sym}_0(m)$. Since $\mathrm{Trace}\, AX{}^tA = \mathrm{Trace}\, X$ for $A \in SO(m)$ and $X \in \mathrm{Sym}_0(m)$, the adjoint mapping $Ad : SO(m) \longrightarrow \mathrm{Iso}(\mathrm{Sym}_0(m))$ can be defined by $AD(A)(X) = AX{}^tA$ for $A \in SO(m)$ and $X \in \mathrm{Sym}_0(m)$. For $p = \frac{m(m+1)}{2} - 1$, let $\Sigma = \{\sigma_1,\dots\sigma_p\}$ be an orthonormal basis of $\mathrm{Sym}_0(m)$. Then we have an isometry $\phi_\Sigma : \mathrm{Sym}_0(m) \longrightarrow \mathbb{R}^p$. It follows that we have a Lie group representation $\rho_\Sigma : SU(m) \longrightarrow SO(p)$ defined by $\rho_\Sigma(A)(y) = \phi_\Sigma \circ Ad(A) \circ \phi_\Sigma^{-1}(y)$ for any $A \in SO(m)$ and $y \in \mathbb{R}^p$. We also have the following proposition.

Proposition 3.5 *Let* $\Sigma = \{\sigma_1, \dots \sigma_p\}$ *be an orthonormal basis of* $\mathrm{Sym}_0(m)$ *with respect to* $(,)$, *where* $p = \frac{m(m+1)}{2} - 1$. *Then* $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathrm{Sym}_0(m), O)$ *are* $SO(m)$*-equivalent if and only if* $\phi_\Sigma \circ f, \phi_\Sigma \circ g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ *are* $\mathcal{K}[\rho_\Sigma(SO(m))]$*-equivalent.*

In the case when $m = 2$, $p = 2$. Then we have

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}.$$

We consider $\delta_1 = \overline{\sigma}_1/\sqrt{2}$ and $\delta_2 = \overline{\sigma_3}/\sqrt{2}$, then $\Sigma = \{\delta_1, \delta_2\}$ is an orthonormal basis of $\mathrm{Sym}_0(2)$. In this case we have

$$A\delta_1 {}^t\!A = \cos 2\theta\delta_1 - \sin 2\theta\delta_1, \quad A\delta_2 {}^t\!A = \sin 2\theta\delta_1 + \cos 2\theta\delta_1,$$

for $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Therefore, the representation matrix of $Ad(A)$ with respect to Σ is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \in SO(2)$. Moreover, for any $B = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, we choose $\theta = -\phi/2$, so that $\rho_\Sigma(A) = B$. This means that $\rho_\Sigma(SO(2)) = SO(2)$. In this case a map germ $f : (\mathbb{R}^n, 0) \longrightarrow \mathrm{Sym}_0(2)$ is also considered to be a quantum mechanical Hamiltonian matrices, which is called a *Type I* in [9]. In this case the matrix valued map germ $f : (\mathbb{R}^n, 0) \longrightarrow \mathrm{Sym}_0(2)$ has the form

$$f(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} f_2(x) & f_1(x) \\ f_1(x) & -f_2(x) \end{pmatrix},$$

so that we have $f(x) = f_1(x)\delta_1 + f_2(x)\delta_3$. Therefore, we have $\phi_\Sigma \circ f(x) = (f_1(x), f_2(x)) \in \mathbb{R}^2$. Then we say that $f, g : (\mathbb{R}^n, 0) \longrightarrow \mathrm{Sym}_0(2)$ are $SO(2)$*-equivalent* if there exist a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and a map germ $A : (\mathbb{R}^n, 0) \longrightarrow SO(2)$ such that $g \circ \phi(x) = A(x) f(x) {}^t\!A(x)$ for any $x \in (\mathbb{R}^n, 0)$. Then we have the following proposition.

Proposition 3.6 *Let* $f, g : (\mathbb{R}^n, 0) \longrightarrow \mathrm{Sym}_0(2)$ *are map germs. Then* f, g *are* $SO(2)$*-equivalent if and only if* $\phi_\Sigma \circ f, \phi_\Sigma \circ g$ *are* $\mathcal{K}[SO(2)]$*-equivalent.*

Let us consider

$$f(x_1, x_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} x_2 & x_1 \\ x_1 & -x_2 \end{pmatrix}.$$

In this case the energy functions (i.e. the eigen value functions) are $E_\pm(x_1, x_2) = \pm\sqrt{x_1^2 + x_2^2}$, so that the graphs of $E_\pm(x_1, x_2)$ form a cone with the vertex at the origin. This a special case of quantum mechanical Hamiltonian of the Dirac equation for massless Dirac Fermions. In this case this cone is called a *Dirac cone* and the origin is called a *Dirac point*. The Dirac point plays an important role in the theory of topological insulaters and the theory of photo-chemical reaction controls. In [11] we give a classification of $f : (\mathbb{R}^n, 0) \longrightarrow \mathrm{Sym}_0(2)$ by $SO(2)$-equivalence for lower codimensions.

4 Examples of $\mathcal{A}[\rho(G)]$-equivalence

Example 4.1 (Divergent diagrams of function-germs) $G = D^*(p)$.

In this case $f = (f_1, \dots f_p) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^2, 0)$ is considered to be a divergent diagram of function-germs

$$\begin{tikzcd}
 & (\mathbb{R}, 0) \\
 & \vdots \\
(\mathbb{R}^n, 0) \arrow[ruu, "f_1"] \arrow[r, "f_i"] \arrow[rdd, "f_p"'] & (\mathbb{R}, 0) \\
 & \vdots \\
 & (\mathbb{R}, 0)
\end{tikzcd}$$

Then $\mathcal{A}[D^*(p)]$-equivalence is considered to be the isomorphism among divergent diagrams of function-germs. This is not a geometric subgroup of $\mathcal{A}$ in the sense of Damon [1]. In particular, the divergent diagrams of function germs for $p = 2$ was classified by formal diffeomorphism-germs in [14].

Example 4.2 (Strict equivalence among divergent diagram)

$$G = \{1\} \oplus GL(p-1, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in GL(p-1, \mathbb{R}) \right\}.$$

We can show that f, g are $\mathcal{A}[\{1\} \oplus GL(p-1, \mathbb{R}))]$-equivalent if and only if there exist diffeomorphism-germs $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}^{p-1}, 0) \longrightarrow (\mathbb{R}^{p-1}, 0)$ such that $f_1(x) = g_1 \circ \phi(x)$ and

$$\psi((f_2(x), \dots, f_p(x))) = (g_2 \circ \phi(x), \dots, g_p \circ \phi(x)).$$

This is a geometric subgroup of $\mathcal{A}$ in the sense of Damon [1]. However, a functional moduli appeared for very low dimensions (cf. [4]). In order to avoid functional modulus, we consider $G = (1^+, GL(p-1, \mathbb{R}))$. Then f, g are $\mathcal{A}[(1^+, GL(p-1, \mathbb{R}))]$-equivalent if and only if there exist diffeomorphism-germs $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$, $\psi : (\mathbb{R}^{p-1}, 0) \longrightarrow (\mathbb{R}^{p-1}, 0)$ and a function-germ $\alpha : (\mathbb{R}^{p-1}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $f_1(x) + \alpha(f_2(x), \dots f_p(x)) = g_1 \circ \phi(x)$ and

$$\psi((f_2(x), \dots, f_p(x))) = (g_2 \circ \phi(x), \dots, g_p \circ \phi(x)).$$

In [4] a generic classification of $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ with respect to $\mathcal{A}[\{1\} \oplus GL(2, \mathbb{R})]$-equivalence is given.

Example 4.3 (Volume preserving diffeomorphisms) $G = SL(p, \mathbb{R})$.

We consider the special linear group $SL(p, \mathbb{R})$. In this case $\mathfrak{sl}(p)$ is the Lie algebra of traceless $p \times p$-matrices. A diffeomorphism germ $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$ with $J_\psi(y) \in SL(p, \mathbb{R})$ for any $y \in (\mathbb{R}^p, 0)$ is a volume preserving diffeomorphism germ. We can show that $\mathcal{A}[SL(p, \mathbb{R})]$ are not geometric subgroups of $\mathcal{A}$ in the sense of Damon [1]. However, the group $SL(p, \mathbb{R})$ is big enough to have nice geometric properties (cf. [3]).

Example 4.4 (Isometries) $G = SO(p)$.

We can show that $f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ are $\mathcal{A}(SO(p)$-equivalent if and only if there exists a diffeomorphism germ $\phi : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ and an isometry $\psi : (\mathbb{R}^p, 0) \longrightarrow (\mathbb{R}^p, 0)$ such that $f \circ \phi = \psi \circ g$. For $n = 2, p = 3$, this equivalence was used in [5, 13, 18] for the study of differential geometry of singular surfaces in $\mathbb{R}^3$. This is not a geometric subgroup of $\mathcal{A}$ in [1]. However, it iduces important geometric invariants for sigular surfaces.

A $\mathcal{A}^*[\rho(G)]$-equivalence

In this appendix we consider $\mathcal{A}^*[\rho(G)]$-equivalence. Here, we use the notations and definitions given by Mather [15].

For a map germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, we define $\omega f : \theta(p) \longrightarrow \theta(f)$ by $\omega f(\eta) = \eta \circ f$. We define $\mathcal{E}_p$-submodules of $\theta(p)$ by

$$\theta^*[\rho(G)]_0(p) = \left\{ \sum_{i=1}^{p}\sum_{j=1}^{p}(\eta_{ij}(y)y_j)\frac{\partial}{\partial y_i} \mid (\eta_{ij}(y)) \in d\rho_e(\mathfrak{g}) \text{ for } y \in (\mathbb{R}^p, 0) \right\}.$$

We also define an $\mathbb{R}$-vector subspace $\theta^*[\rho(G)](p) = \mathbb{R}^p \oplus \theta^*[\rho(G)]_0(p)$ of $\theta(p)$. Then we can show that $(tf, \omega f, \theta(n), \theta^*[\rho(G)]_0(p), \theta(f))$ is a mixed homomorphism of finite type over $f^* : \mathcal{E}_p \longrightarrow \mathcal{E}_n$ in the sense of Mather [15]. We denote that

$$T_e\mathcal{L}^*[\rho(G)](f) = \omega f(\theta^*[\rho(G)](p)),\ T\mathcal{L}^*[\rho(G)](f) = \omega f(\theta^*[\rho(G)]_0(p)),$$

$$T_e\mathcal{A}^*[\rho(G)](f) = tf(\theta(n)) + T_e\mathcal{L}[\rho(G)](f) \text{ and } T\mathcal{A}^*[\rho(G)](f) = tf(\mathfrak{M}_n\theta(n)) + T\mathcal{L}^*[\rho(G)](f).$$

In this case, $\mathcal{A}^*[\rho(G)]$-equivalence for any $\rho : G \longrightarrow GL(p, \mathbb{R})$ induces a geometric subgroup of $\mathcal{A}$ in the sense of Damon [1].

Example A.1 ($G = SO(p)$) We now consider $p = 2$. In this case we have

$$\theta^*[SO(2)]_0(2) = \left\{ \sum_{i=1}^{2}\sum_{j=1}^{2}(\eta_{ij}(y)y_j)\frac{\partial}{\partial y_i} \mid \begin{pmatrix} \eta_{11}(y) & \eta_{12}(y) \\ \eta_{21}(y) & \eta_{22}(y) \end{pmatrix} \in \mathfrak{so}(2)(\mathcal{E}_2) \text{ for } y \in (\mathbb{R}^p, 0) \right\}.$$

Since $\mathfrak{so}(2)$ is the Lie algebra of anti-symmetric matrices,

$$\begin{pmatrix} \eta_{11}(y) & \eta_{12}(y) \\ \eta_{21}(y) & \eta_{22}(y) \end{pmatrix} \in \mathfrak{so}(2)(\mathcal{E}_2) \text{ if and only if } \eta_{11}(y) = \eta_{22}(y) = 0, \eta_{12}(y) = -\eta_{21}(y).$$

It follows that

$$\theta^*[SO(2)]_0(2) = \left\{ \eta(y)\left(y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} \right) \mid \eta(y) \in \mathcal{E}_2 \right\}.$$

For $f = (f_1, f_2) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^2, 0)$, we have

$$T\mathcal{L}^*[SO(2)](f) = \omega f(\theta[SO(2)]_0(2)) = \left\langle \left(f_2\frac{\partial}{\partial y_1} \circ f - f_1\frac{\partial}{\partial y_2} \circ f \right) \right\rangle_{f^*(\mathcal{E}_2)}$$

and

$$T_e\mathcal{L}^*[SO(2)](f) = \mathbb{R}^2 \oplus \left\langle \left(f_2 \frac{\partial}{\partial y_1} \circ f - f_1 \frac{\partial}{\partial y_2} \circ f \right) \right\rangle_{f^*(\mathcal{E}_2)}.$$

On the other hand, let $f = (f_1, \dots, f_p) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be a map germ. By the similar arguments to the above case, we have

$$T\mathcal{L}^*[SO(p)](f) = \left\langle \left\{ \left(f_j \frac{\partial}{\partial y_i} \circ f - f_i \frac{\partial}{\partial y_j} \circ f \right) \;\middle|\; 1 \leq i < j \leq p \right\} \right\rangle_{f^*(\mathcal{E}_p)}$$

and

$$T_e\mathcal{L}^*[SO(p)](f) = \mathbb{R}^p \oplus \left\langle \left\{ \left(f_j \frac{\partial}{\partial y_i} \circ f - f_i \frac{\partial}{\partial y_j} \circ f \right) \;\middle|\; 1 \leq i < j \leq p \right\} \right\rangle_{f^*(\mathcal{E}_p)}.$$

By definition, $\mathcal{A}^*[\rho(G)]$-equivalence among map-germs implies $\mathcal{K}[\rho(G)]$-equivalence. However, we have the following problems:

(1) Are there intersting examples?

(2) What is a geometric meaning of $\mathcal{A}^*[\rho(G)]$-equivalence?

参考文献

[1] J. Damon, The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$. Memoirs of A.M.S. **50** No. **306**, (1984)

[2] A. Dimca, Function germs on isolated hypersurface singularities. Compositio Math. **53**, 245–268, (1984)

[3] W. Dormitrz and J. H. Rieger, Volumte preserving subgroups of $\mathcal{A}$ and $\mathcal{K}$ and singularities in unimodular geometry. Math. Ann. **345**, 783–817, (2009)

[4] J. P.Dufour, Families de courses planes differentiables. Topology, **22**, 449–474, (1983)

[5] T. Fukui and M. Hasegawa, Fronts of Whitney umbrella- a differential geometric approach via blowing up. J. Singul. **4**, 35–67, (2012)

[6] J. J. Gervais, Suffuciency of jets. Pacific Journal of Mathematics, **72**, 419–422, (1977)

[7] J. J. Gervais, G-stability of mappings and stability of bifurcation diagrams. J. London Math. Soc. **25**, 551–563, (1982)

[8] J. J. Gervais, Deformations G-verselles et G-stables. Can. J. Math. **XXXVI**, 9–21, (1984)

[9] G. A. Hagedorn, Molecular propagation through electoron energy level crossings. Mem. Amer. Math. Soc. **111**, 130pp (1994)

[10] S. Izumiya and S. Matsuoka, Notes on smooth function germs on varieties. Proc. Amer. Math. Soc. **97**, 146–150, (1986)

[11] S. Izumiya, M. Takahashi and H. Teramoto, Geoemtric equivalence among smooth section germs of vector bundles with respect to structure groups. preprint (2017).

[12] H. B. Lawson and M. L. Michelsohn, Spin Geometry, Princeton Univ. Press, Princeton, (1989).

[13] L. Martins and K. Saji, Geometric invariants of cuspidal edges. Canadian Journal of Mathematics, to appear

[14] S. Mancini, M. A. S. Ruas and M. A. Texeira, On divergent diagrams of finite codimendion. PORTUGALIAE MATHEMATCA, **59**, 179–194, (2002)

[15] J. Mather, Stability of C^∞-Mappings III. Finitely Determined Map Germs. Pibl. Math. I.H.E.S. **36** , 127–156, (1968)

[16] H. Teramoto, K. Kondo, S. Izumiya, M. Toda and T. Komatsuzaki, Classification of Hamiltonians in neighborhoods of band crossings in terms of the theory of singularities. In preparation (2016)

[17] J. C. Tougeron, *Ideaux fonctions differentiables.* Springer-Verlag, Berlin, Heiderberg, New York 1972

[18] J. M. West, The differential geometry of the cross-cap. Ph.D. thesis, The University of Liverpool, 1995.

[19] H. Weyl, *The classical groups, their invariants and representations.* Princeton Univ. Press, (1939)